ON THE POSSIBILITY OF REALIZING A NONHOLONOMIC CONSTRAINT BY MEANS OF VISCOUS FRICTION FORCES

(O VOZMOZHNOSTI REALIZATSII NEGOLONOMNOI SVIAZI POSREDSTVOM SIL VIAZKOGO TRENIIA)

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It is shown by the example of "Chaplygin's sled" that a nonholonomic constraint may be realized by means of a viscous friction force in the limiting case, when the coefficient of viscous friction is equal to infinity. The result thus obtained disproves Carathéodory's conclusion that such realization is impossible.

1. In 1933 Carathéodory [1] investigated the inertial motion of Chaplygin's sled [2] in particular case in which the center of mass of the system lies on a straight line passing through the plane of the runner. In this case the motion of the sled is described by the differential equations

$$u^{\cdot} = a\omega^2, \quad ak^2\omega^{\cdot} = u\omega \quad (k^2 = 1 + J/ma^2) \tag{1.1}$$

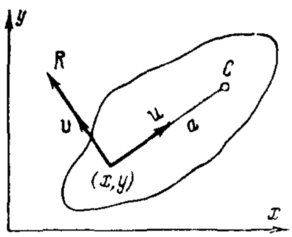

Fig. 1

Here u is the magnitude of the velocity of the point of contact between the runner and the plane, ω is the angular velocity of rotation of the sled, J is the central moment of inertia, m is the mass, a is the distance between the point (x, y) of contact between runner and plane and the center of mass C of the sled (Fig. 1). The reaction force R resists any sliding of the sled in a direction perpendicular to the plane of the runner, and therefore in the ordinary motion of the sled described by Equations (1.1) we have the velocity component $v = 0$. Assuming that R has the nature of a viscous friction force, Carathéodory represented this force in the form

$$R = -Nv \tag{1.2}$$

where N is the coefficient of viscous friction, a very large number. If we introduce the small parameter $\varepsilon = J/Na^2$, then for $v \neq 0$ the equations of motion of the sled can be written in the form

$$v = \varepsilon a\omega^{\cdot}, \qquad u^{\cdot} = a\omega^2 + \varepsilon a\omega\omega^{\cdot}; \qquad ak^2\omega^{\cdot} + u\omega = -\varepsilon a\omega^{\cdot\cdot} \tag{1.3}$$

From Equations (1.3) it can be seen at once that when $\varepsilon = 0$, they reduce to Equations (1.1). Furthermore, Carathéodory reasoned as follows: we consider the motion of the sled for the initial conditions $u_0 = 0$, $v_0 = 0$, $\omega_0 = kc$, where $c \neq 0$ is a certain constant. Then, for $\omega^{\cdot}$ and $\omega^{\cdot\cdot}$, from Equations (1.3), we obtain the initial values $\omega_0^{\cdot} = 0$, $\omega_0^{\cdot\cdot} = 0$, and from Equations (1.1) we find the values $\omega_0^{\cdot}=0$, $\omega_0^{\cdot\cdot}=-kc^3$. From this Carathéodory drew the conclusion that for any arbitrarily small $\epsilon \neq 0$ the trajectory of the sled differs from the sled trajectory for $\epsilon = 0$ and that consequently the nonholonomic constraint under consideration cannot be realized by means of viscous friction forces.

2. Caratheodory's reasoning cannot be accepted as convincing because when we compare the motions described by Equations (1.1) and (1.3), we must bear in mind the fact that the system of equations (1.3) describes the motion of the representative point in three-dimensional phase space, while the system of equations (1.1) describes it in two-dimensional space. Consequently, when we pass from (1.3) to (1.1), the system degenerates. Let us again examine the system of equations (1.3). We introduce the small parameter $\mu = \epsilon a$ and the variable $\sigma = \omega^{\cdot}$; then Equations (1.3) can be written in the form of three first-order equations

$$u^{\cdot} = a\omega^2 + \mu\omega\sigma, \qquad \omega^{\cdot} = \sigma, \qquad \mu\sigma^{\cdot} = -ak^2\sigma - u\omega \quad (2.1)$$

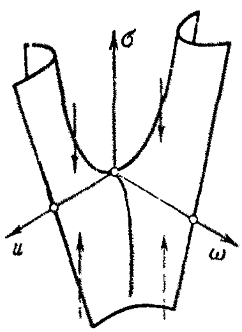

Fig. 2

which describe the motion of the representative point $u\omega\sigma$-space, and the first equation of (1.3) becomes the simple relationship $v = \mu\sigma$, connecting the variables v and σ . From the last equation of the system (2.1) it follows that as $\mu \to 0$ the $u\omega\sigma$ space becomes a region of rapid motions (with respect to the coordinate σ) except for the surface

$$ak^2\sigma + u\omega = 0 \qquad (2.2)$$

which is found to be a region of slow motions, and for fast motions it is a region of stable equilibrium states (Fig.2). As $\mu \to 0$, the representative point passes from any point of $u\omega\sigma$ space to the surface (2.2) with a velocity

$$\sigma^{\cdot} = -\lim_{\mu\to 0} \frac{ak^2\sigma + u\omega}{\mu} = \begin{cases} -\infty, & \text{when } ak^2\sigma + u\omega > 0 \\ +\infty, & \text{when } ak^2\sigma + u\omega < 0 \end{cases}$$

and thereafter moves on this surface in accordance with Equations (1.1), which are obtained from (2.1) for the case $\mu = 0$. Equations (1.1) describe a slow motion of the representative point on the surface (2.2) projected onto the $u\omega$ plane. Dividing the first equation of (1.1) by the second one, we obtain the differential equation

$$\frac{du}{d\omega} = -a^2k^2\frac{\omega}{u}$$

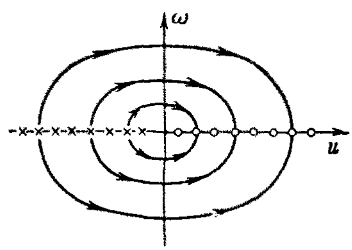

Fig. 3

From this we find a family of integral curves $u^2 + a^2k^2\omega^2 = \text{const}$, which divide the $u\omega$ plane into trajectories (Fig.3). The line $\omega = 0$ contains the equilibrium states of the system (1.1); unstable equilibrium states are found on the half-line ($\omega = 0$, $u < 0$) and stable equilibrium states on the half-line ($\omega = 0$, $u > 0$). The points on the axis $\omega = 0$ correspond to rectilinear motion of the sled at the constant velocity $u = u_0 = \text{const}$. This motion of the sled will be unstable if the runner is situated in front of the center of mass and will be stable if the runner is situated behind the center of mass. When the representative point moves along the arc of an ellipse, the sled runner describes one of the beak-shaped curves for which an analytic expression was found by Carathéodory.

In the case of small values of the parameter $\mu (\mu \neq 0)$ the representative point in $u\omega\sigma$ space passes from points outside a finite neighborhood of the surface (2.2) into a μ neighborhood of this surface in a time of the order of $\mu \ln \mu^{-1}$ and thereafter remains in the neighborhood [3].

Let us now turn to Carathéodory's reasoning. First of all, we note that for Carathéodory's initial conditions ($u_0 = 0$, $\omega_0 = kc$, $\sigma_0 = 0$), the representative point in $u\omega\sigma$ space is situated on the surface (2.2) and consequently in the region of slow motion, that is represented by a μ-neighborhood of the surface (2.2), when $\mu \neq 0$ ($\mu < 1$). According to the foregoing, the representative point will remain in this region thereafter as well. Let us study further the behavior of the function $\sigma^{\cdot} = \sigma^{\cdot}(t)$. For this purpose, we shall consider the motion of the representative point in $u\omega\sigma^{\cdot}$ space. Differentiating the last equation (2.1) and eliminating the variable σ, we obtain from (2.1) the system of differential equations

$$ak^2 \frac{du}{dt} = a^2k^2\omega^2 - \mu\omega\,(u\omega + \mu\sigma^{\cdot})\,, \qquad ak^2 \frac{d\omega}{dt} = -\,u\omega - \mu\sigma^{\cdot} \tag{2.3}$$

$$\mu a k^2 \frac{d\sigma^{\cdot}}{dt} = [\mu\,(u + \mu\omega^2) - a^2k^4]\,\sigma^{\cdot} - a^2k^2\omega^3 + \omega u^2 + \mu u\omega^3$$

describing the motion of the representative point in $u\omega\sigma^{\cdot}$ space. From the last equation of (2.3) it follows that as $\mu \to 0$, the $u\omega\sigma^{\cdot}$ space becomes a region of fast motions (with respect to the coordinate $\sigma^{\cdot}$) except for the surface

$$a^2k^4\sigma^{\cdot} = \omega u^2 - a^2k^2\omega^3 \tag{2.4}$$

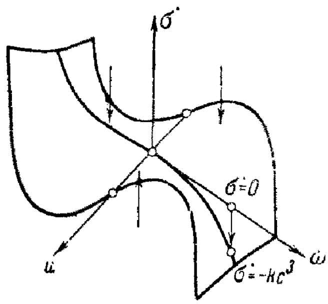

Fig. 4

which is found to be a region of slow motions (Fig.4). These slow motions will be stable with respect to fast motions. In the case of small values of the parameter μ ($\mu \neq 0$) the region of slow motions will be a μ-neighbohood of the surface

$$[\mu\,(u + \mu\omega^2) - a^2k^4]\,\sigma^{\cdot} - a^2k^2\omega^3 + \omega u^2 + \mu u\omega^3 = 0 \tag{2.5}$$

In the restricted region of $u\omega\sigma^{\cdot}$ space the surface (2.5) differs from (2.4) by a small quantity of the order of μ , and as $\mu \to 0$, the region approaches the surface (2.4). For Carathéodory's initial conditions ($u_0 = 0$, $\omega_0 = kc$, $\sigma_0^{\cdot} = 0$) the representative point in (u, ω, $\sigma^{\cdot}$) space lies outside a finite neighborhood of the surface (2.5) and, in accordance with the foregoing, it will enter a μ neighborhood of this surface in a time of the order of $\mu \ln \mu^{-1}$. For the values $u = 0$, $\omega = kc$, on the surface (2.4) we have $\sigma^{\cdot} = -kc^3$. Consequently, as $\mu \to 0$, the change in $\sigma^{\cdot}$ from $\sigma^{\cdot} = 0$ to $\sigma^{\cdot} = -kc^3$ becomes an instanteneous jump (see Fig.4). Thus, the contradiction reached by Carathéodory is resolved as follows: although the values of $\omega^{\cdot\cdot} = \sigma^{\cdot}$ at the initial instant of time $t = 0$ are actually different in the cases $\mu = 0$ and $\mu \neq 0$ ($\mu \ll 1$), this difference nevertheless vanishes within a period of the order of $\mu \ln \mu^{-1}$ As $\mu \to 0$, the region of slow motions is pressed toward the surface, and the limiting (slow) motion of the nondegenerate system (2.1) coincides with the motion of the limiting (degenerate) system (1.1). The slow motion is found to be stable with respect to the fast motions.

The above discussion leads to the conclusion that a nonholonomic constraint arising in the motion of the runner on the plane may be realized by means of viscous friction forces and corresponds to the case in which the coefficient of viscous friction is equal to infinity.

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